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The Target Rate and Term Structure of Interest Rates

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Abstract

This paper presents a tractable bond valuation model, which further develops the approach proposed by Piazzesi (2005). The short term inter-bank interest rate is equal to the target rate set by the central bank plus a spread. Bond yields are driven by the intensities that determine the probabilities that the central bank may raise or cut the target interest rate. Unlike in Piazzesi (2005), negative intensities have a convenient interpretation and do not complicate estimation, and two accurate approximations to the bond pricing equation provide new closed form solutions for discount bond prices that require no numerical integration. Unlike in Piazzesi the target interest rate can be constrained to be non-negative. Yields, especially long term ones, decrease when the central bank is expected to decide more frequent and/or larger average future changes in the target interest rate. The model lends itself to easy calibration and estimation.

Key words: bond valuation; target interest rate; closed form solution; yield curve; central banker's meetings.

JEL classification: G13.

1 Introduction and literature

The term structure of inter-bank interest rates, especially the short end of it, is driven by the central bank current and expected future policy that sets the "target" interest rate (in the case of the FED) or the "reference" interest rate (in the case of the ECB). This paper presents a bond pricing model can accounts for this stylised fact. For simplicity hereafter we just refer to the "target" interest rate. More specifically this paper further develops the innovative approach proposed by Piazzesi (2005). Piazzesi put forward an affine term structure model whose corner stone is the observable FED target interest rate and the market perceived probabilities that the FED may alter the target interest. This paper

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builds on this same corner stone, strong of the fact that Piazzesi showed that "target (rate) data improve the fit of the yield curve model" and especially the fit to yields with maturity up to the two years. The use of the observed target rate for pricing bonds distinguishes the paper by Piazzesi as well as this paper from the rest of the default-free bond pricing literature, such as for example Vasicek (1977), Langetieg (1980), Cox, Ingersoll and Ross (1985), Sun (1992), Constantinides (1992), Longstaff and Schwartz (1992), Duffie and Kan (1996), Bansal and Zhou (2002), Duffie, Filipovic and Schachermayer (2003), Gourieroux, Monfort and Polimenis (2002), Dai and Singleton (2000, 2002, 2003), etc.. .

The models proposed in this paper can also be viewed as tractable special cases of the general affine term structure model specification of Duffie and Kan (1996). Unlike in Piazzesi, this paper provides closed form solutions for discount bonds that require no numerical integration and that rely on accurate approximations to the bond pricing equations. Notably such closed form solutions are available both under the assumption that the target rate may change at any time and under the assumption that the target rate may just change on central bankers' scheduled meeting dates. The former of these assumptions can be considered as an approximation to the latter. The bond prices mainly depend on the target interest rate and a latent factor driving the central bank target setting policy. Unlike in Piazzesi, in this paper the target interest rate can be constrained to be non-negative, which is consistent with historical experience. In Piazzesi the intensities that drive the probabilities of changes in the target rate can be negative and this complicates the estimation of model parameters. In this paper negative intensities have a convenient economic interpretation and do not complicate estimation.

The model offers the flexibility to match yield curves of a variety of shapes. Yields, especially long term ones, are driven by "volatility kicker" effects. Yields decrease in the volatility of the latent factor that drives changes in the target rate. Similarly yields, especially long term ones, decrease as future changes in the target interest rate increase in expected frequency and magnitude, i.e. as the target rate becomes more volatile. Finally frequent small changes in the target rate have virtually the same effect on yields as less frequent larger changes.

The paper is organised as follows. The next section introduces the bond pricing model in the two settings where the target rate can change at any time and where it can only change on set dates. Then further results under these two settings are presented for a model variant that constrains the target rate to be non-negative. Finally, after a brief discussions of model calibration and estimation, the conclusions follow.

2 The bond pricing model

This section presents the theoretical bond pricing model in a setting where the target interest rate may theoretically turn negative, although this may be unlikely. This shortcoming affects also the similar model proposed by Piazzesi

(2005). Later on we will look at a model variant where the shortcoming of a possible negative target rate is overcome.

We assume that the default free short interest rate is

$$r = s + x \tag{1}$$

where x is the target rate set by the central bank and s is the spread between r and x. s follows the risk-neutral process

$$ds = -asdt + \sigma dz \tag{2}$$

where a and σ are constant and dz is the differential of a Wiener process. During any infinitesimal time interval dt, there is a risk-neutral probability λdt that the central bank will move the target rate x. λ represents the risk-neutral intensity that drives the probability of a change in the target rate x. x can change into $x+\pi$, where $\pi > 0$ is a constant, typically equal to 0.0025 or 25 basis points. IN reality π would be stochastic and could assume values equal to 0.005 or larger. This could be accommodated in the models here presented since it does not scupper their tractability, but for expositional simplicity hereafter we assume $\pi = 0.0025$.

We set the intensity

$$\lambda = k\left(y - x - s\right) \tag{3}$$

where y is a latent factor that follows the risk-neutral process

$$dy = b\left(m - y\right)dt + qdz_y. \tag{4}$$

b, m and q are constant, while dz_y is the differential of the Wiener process driving dy and such that $dz \cdot dz_y = \rho dt$. We can think of y as the driver of the central bank target rate decisions over time. The equations for λ and dy imply that λ may well be negative, which is a desirable feature rather than a drawback. As will be explained shortly, when $y < x + s$, then $\lambda < 0$ and the target rate may drop from x to $x - \pi$ rather than rise from x to $x + \pi$. Notice that, although y is a latent factor, we well be able to inferred from it from observed bond yields.

Then the absence of arbitrage opportunities implies that the value V of a default-free zero discount bond must satisfy the equation

$$\frac{\partial V}{\partial t}+\frac{\partial^2 V}{\partial s \partial y}\rho q \sigma+\frac{\partial^2 V}{\partial s^2}\frac{1}{2}\sigma^2+\frac{\partial^2 V}{\partial y^2}\frac{1}{2}q^2-\frac{\partial V}{\partial s}as+\frac{\partial V}{\partial y}b\left(m-y\right)-\left(s+x\right)V+k\left(y-x-s\right)\left(V_\pi - V\right)=0 \tag{5}$$

subject to the terminal condition $V\left(T\right) = 1$ where T is the bond maturity date. V denotes the bond value if the target rate is x, whereas V_π denotes the bond value if the target rate changes to $x + \pi$. The solution to the above pricing equation is such that

$$V = e^{A+Bx+Ds+Cy}, V_\pi = e^{A+B(x+\pi)+Ds+Cy} \tag{6}$$

where A, B, D and C a function of time t and are the solution to the following system of ordinary differential equations (ODE's)

$$\frac{\partial A}{\partial t} + DC\rho q\sigma + D^2 \frac{1}{2}\sigma^2 + C^2 \frac{1}{2} q^2 + Cbm = 0 \tag{7}$$
$$\frac{\partial B}{\partial t} - 1 - k\left(e^{B\pi} - 1\right) = 0 \tag{8}$$
$$\frac{\partial D}{\partial t} - Da - 1 - k\left(e^{B\pi} - 1\right) = 0 \tag{9}$$
$$\frac{\partial C}{\partial t} - Cb + k\left(e^{B\pi} - 1\right) = 0. \tag{10}$$

subject to the terminal conditions $A(T) = B(T) = C(T) = D(T) = 0$. $A(T)$, $B(T)$, $C(T)$, $D(T)$ denote the terminal values of the functions. This system of ODE's can be quickly solved numerically, for example by employing the Euler numerical scheme or one of its variants.

The central feature of the model is that the target rate x may change at any time during the bond life. We can write the expected change in bond value due to a change in x during any infinitesimal interval dt as

$$\begin{aligned} k(y - x - s) \cdot dt \cdot (V_\pi - V) &= k(y - x - s) \cdot dt \cdot \left(e^{A+B(x+\pi)+Ds+Cy} - e^{A+Bx+Ds+Cy}\right) \\ &= k(y - x - s) \cdot dt \cdot e^{A+Bx+Ds+Cy}\left(e^{B\pi} - 1\right) \\ &\backsimeq k(y - x - s) \cdot dt \cdot e^{A+Bx+Ds+Cy} B\pi \\ &= -k(y - x - s) \cdot dt \cdot e^{A+Bx+Ds+Cy}(-B\pi). \end{aligned} \tag{11}$$

The right hand side of the equation on the third line approximates the second line. The approximation is

$$e^{B\pi} \backsimeq 1 + B\pi. \tag{12}$$

This approximation is quite accurate: the smaller the absolute values of B and π, the more accurate it is. Since B rises with the bond maturity, the approximation is more accurate for short term bonds. The fourth line simply re-expresses the third line through a double change of sign, which highlights the following. When $\lambda = k(y - x - s) > 0$, λdt denotes the probability a change from x to $x + \pi$ during dt, which entails a loss to bondholders since $\pi > 0$ and $B \leq 0$. When $\lambda = k(y - x - s) < 0$, $-\lambda dt$ denotes the probability a change from x to $x - \pi$ during dt, which entails a gain to bondholders. Also in Piazzesi the intensities that drive the probabilities of changes in the target rate can be negative, but this feature the estimation of parameters in her model. In this paper negative intensities have a convenient economic interpretation, as just explained, so that estimation is not hindered by the requirement that intensities be positive. This simplifies parameter estimation. We may think of y as the level toward which the central bank tends to drive $r = x + s$, in which case y would depend on the central bank policy in response to the evolution of the macro-economy. When r is smaller (greater) than y, the central bank tends to raise (cut) the target rate by the amount π. A problem with this model, and one that also hinders the model proposed by Piazzesi (2005), is that the latent factor y, the target rate x and the inter-bank short rate r can turn negative, although, especially for the target rate x, this may not very very likely. Later we will see how to overcome this problem.

2.1 Approximation and closed form solution

The system of ODE's 7, 8, 9, 10 can be solved numerically, but insightful closed form solutions for A, B, C and D are possible if we employ approximation 12. This approximation entails little loss in accuracy, as will be shown, and it implies that the ODE system satisfied by A, B, C and D can be approximated by the following system

$$\frac{\partial A}{\partial t} + DC\rho q\sigma + D^2\frac{1}{2}\sigma^2 + C^2\frac{1}{2}q^2 + Cbm = 0 \tag{13}$$
$$\frac{\partial B}{\partial t} - 1 - kB\pi = 0 \tag{14}$$
$$\frac{\partial D}{\partial t} - Da - 1 - kB\pi = 0 \tag{15}$$
$$\frac{\partial C}{\partial t} - Cb + kB\pi = 0. \tag{16}$$

subject to $A(T) = B(T) = C(T) = D(T) = 0$. The solution to this system is

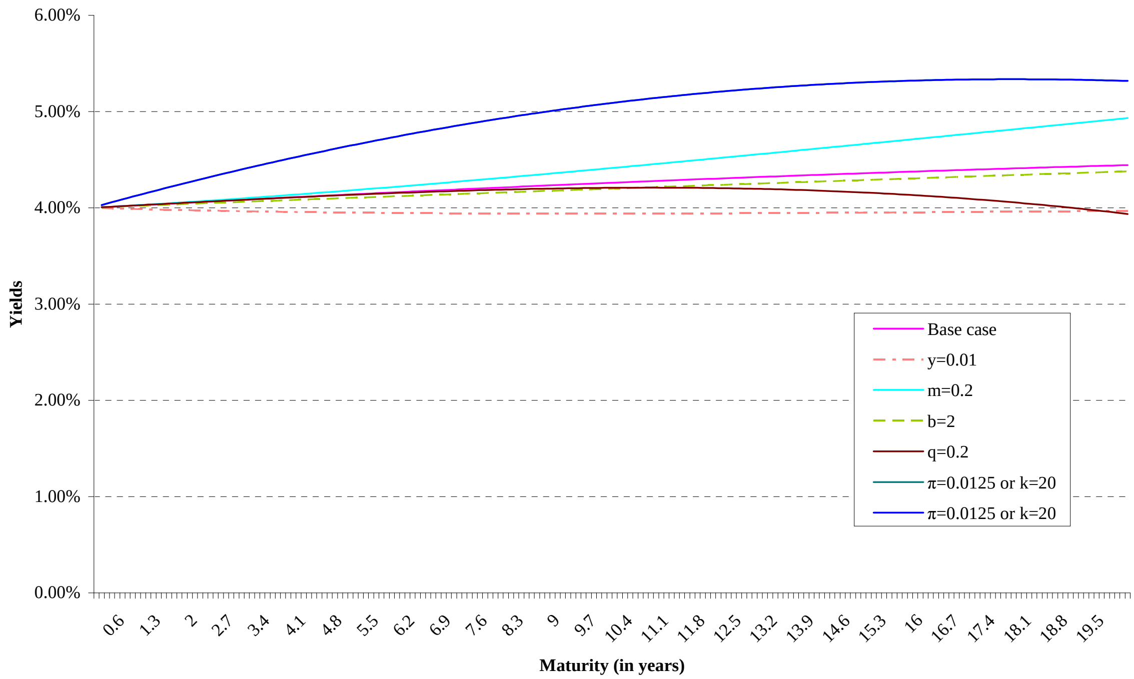
$$B(t) = \frac{e^{-\pi k(T-t)} - 1}{\pi k} \tag{17}$$
$$C(t) = \frac{b\left(e^{-\pi k(T-t)} - 1\right) + \pi k\left(1 - e^{-b(T-t)}\right)}{b^2 - \pi bk} \tag{18}$$
$$D(t) = \frac{e^{-a(T-t)} - e^{-\pi k(T-t)}}{a - \pi k} \tag{19}$$

and the solution for $A(t)$ is provided in the Appendix. To stress the dependence on the time variable t, here and at some points also later in the text, we use the notation $A(t)$, $B(t)$, $C(t)$ and $D(t)$. Notice that these solutions imply the parameter restrictions $\pi k \neq 0, a \neq \pi k, b \neq \pi k$. Now we use this model in some comparative statics.

2.2 Comparative statics

Overall the above model predicts yield curves of a variety of shapes: upward-sloping, downward-sloping, hump-shaped and more. We assume an illustrative base case scenario whereby $s = \rho = \sigma = a = 0, b = 0.1, m = 0.08, q = 0.05, k =$

Figure 1: Yield curves. The legend describes the various scenarios. Scenarios different from the base case differ only because of the different input indicated for each scenario.



$4, \pi = 0.0025$. This scenario effectively regards the factor s as absent, thus highlighting the specificity of the bond pricing model. Results are depicted in Figure 1.

The errors due to the approximation in equation 12 seems negligible. Under realistic parameters, the largest errors are less than 6 basis points for twenty year maturity discount bond yields.

As expected yields rise in the latent factor y and in the mean reversion level m. When $x > y$ ($x < y$) the curve tends to slope downward (upward). The effects of the parameters b, q, k and π on yields, and in particular long term yields, are driven by "volatility kicker" effects: when these parameters change so as to raise the conditional variance of y or x, expected future bond prices as well as current bond prices tend to rise, since bond prices are convex in y and x. Hence when b, q, k and π change so as to increase the volatility of y and x, bond yields, especially long term ones, tend to decrease, all other things being equal. For example, "volatility kicker" effects explain why:

- yields decrease in the volatility of y as measured by q and when q is extremely high long term yields can even turn negative;

- yields, especially long term ones, rise in b, at least when when $x \leq y$; when

$x > y$ yields may decrease in b, because x would get pulled toward y more quickly.

The above formulae imply that bond prices only depend on the product $k\pi$, rather than on π and k separately. It follows that frequent small changes have the same effect on yields as infrequent larger ones. The effect of $k\pi$ on yields is complex. For realistic parameters, i.e. for $k\pi$ equal to about $4 \cdot 0.0025$, yields, especially long term ones, decline in π and k. This effect is again a "volatility kicker" effect: as π and k rise, the conditional variance of x rises, i.e. x becomes more volatile, bond prices rise and yields decrease.

The model has thus far assumed that changes to the target rate x could occur at any time. This assumption is an approximation of the fact that central bankers' meetings are scheduled on set dates and that, if they intend to alters the target rate, they will most likely do so on one of the set meeting dates. Thus we now turn to modelling this stylised fact in greater detail.

2.3 When the target rate can change just on set dates

Now we retain the setting of the above model, but assume that x can change just on one of the dates on which central bankers are scheduled to meet. We denote the sequence of scheduled meeting dates before the bond maturity T as $T_1 < .. < T_i < .. < T_n \leq T$. Hereafter we keep using V to denote the bond value at any time during the life of the bond $[t, T]$, we use V_i to refer specifically to the bond value V during the interval $[T_i, T_{i+1}]$, V_n for the bond value V during the interval $[T_n, T]$ and V_0 for the bond value V during the interval $[t, T_1]$, i.e. $i = 0$ to n.

Then the absence of arbitrage opportunities implies that the value of the discount bond V must satisfy

$$\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial s \partial y}\rho q \sigma + \frac{\partial^2 V}{\partial s^2}\frac{1}{2}\sigma^2 + \frac{\partial^2 V}{\partial y^2}\frac{1}{2}q^2 - \frac{\partial V}{\partial s}as + \frac{\partial V}{\partial y}b\,(m-y) - (s+x)\,V = 0 \tag{20}$$

subject to $V(T) = 1$. Additionally, on any central bankers' meeting date T_i the value of the discount bond must satisfy the continuity condition

$$V_{i-1} = E\,(V_i) = p\,(x, y)\,V_{\pi,i} + (1 - p\,(x, y))\,V_i \tag{21}$$

where $p\,(x, y)$ denotes the risk-neutral probability that the central bank may alter the target rate x at time T_i. At any time $p\,(x, y)$ is a function of the contemporaneous values of y and x. $V_{\pi,i}$ denotes again the value of the bond if the target rate is moved to $x + \pi$, whereas V_i denotes the value of the bond if the target rate remains unaltered. To solve this pricing problem we can approximate condition 21. In fact, if the solution to this pricing problem is again of the type $V = e^{A+Bx+Ds+Cy}$, $V_\pi = e^{A+B(x+\pi)+Ds+Cy}$, then we can approximate continuity condition 21 at T_i as

$$
\begin{aligned}
e^{A_{i-1}+xB_{i-1}+sD_{i-1}+yC_{i-1}} &= p(x,y)\,e^{A_i+(x+\pi)B_i+sD_i+yC_i}+(1-p(x,y))\,e^{A_i+xB_i+sD_i+yC_i} \qquad (22)\\
&\backsimeq p(x,y)(1+B_i\pi)\,e^{A_i+xB_i+sD_i+yC_i}+(1-p(x,y))\,e^{A_i+xB_i+sD_i+yC_i}\\
&= (p(x,y)\,B_i\pi+1)\,e^{A_i+xB_i+sD_i+yC_i}\\
&\backsimeq e^{A_i+(x+p(x,y)\pi)B_i+sD_i+yC_i}
\end{aligned}
$$

where we employ the notations $V_i = e^{A_i+xB_i+sD_i+yC_i}$, $V_{\pi,i} = e^{A_i+(x+\pi)B_i+sD_i+yC_i}$. The approximation shown in the fourth line is again quite accurate and enables us to derive convenient closed form solutions for bond prices. Then we specify the risk-neutral probability that the central bank may change the target interest rate from x to $x+\pi$ on any date T_i as

$$p(x,y) = k(y-x). \qquad (23)$$

Notice that $p(x,y)$ may well be negative. As before, when $p(x,y) < 0$, $-p(x,y)$ should be interpreted as the risk-neutral probability that the target interest rate may move from x to $x-\pi$. Instead, when $p(x,y) > 0$, $p(x,y)$ should be interpreted as the risk-neutral probability that the target interest rate may move from x to $x+\pi$. In this context the parameter k is likely to be proportional to the time between successive central bankers' meetings, i.e. proportional to $T_i - T_{i-1}$. In fact the less frequent the meetings are, the more likely it is that the target rate will change on a meeting date and hence the larger k will be. We notice that we could more generally assume that $p(x,y) = k(y-x-s)$. This more general assumption would not scupper the closed form solutions that we are going to derive, but we do not pursue this for simplicity.

It follows from approximation 22 and from assumption 23 that on any of the meeting dates $T_i < T$ the continuity condition 21 becomes

$$e^{A_{i-1}+xB_{i-1}+sD_{i-1}+yC_{i-1}} \backsimeq e^{A_i+B_i(1-\pi k)x+(C_i+B_i\pi k)y+D_i s}. \qquad (24)$$

With this tractable approximation of the continuity condition, the conjectured solution $V = e^{A+Bx+Cy+Ds}$ satisfies pricing equation 20 subject to its conditions if and only if A_i, B_i, C_i and D_i satisfy the following system for all values of i

$$\frac{\partial A_i}{\partial t} + DC_i\rho q\sigma + \frac{D^2\sigma^2}{2} + C_i^2\frac{1}{2}q^2 + C_i bm = 0 \qquad (25)$$

$$\frac{\partial B_i}{\partial t} - 1 = 0 \qquad (26)$$

$$\frac{\partial D}{\partial t} - Da - 1 = 0 \qquad (27)$$

$$\frac{\partial C_i}{\partial t} - C_i b = 0 \qquad (28)$$

subject to the terminal conditions $A_n(T) = B_n(T) = C_n(T) = D(T) = 0$. Moreover the continuity condition 24 implies in turn the following conditions

$$
\begin{aligned}
A_{i-1}(T_i) &= A_i(T_i) && (29)\\
B_{i-1}(T_i) &= B_i(T_i)(1-\pi k) && (30)\\
C_{i-1}(T_i) &= C_i(T_i)+\pi k B_i(T_i) && (31)
\end{aligned}
$$

for all dates $T_i < T$. We now turn to solving the system of ODE's and conditions. The solution for D is the same as in equation 19 but with $k = 0$, i.e. for all time intervals

$$
D(t) = \frac{e^{-a(T-t)} - 1}{a}. \tag{32}
$$

For any generic interval $T_{i-1} \le t \le T_i$, to determine B_{i-1} we solve $\frac{\partial B_{i-1}}{\partial t} - 1 = 0$, subject to $B_{i-1}(T_i) = B_i(T_i)(1-\pi k)$. The solution for the generic interval $T_{i-1} \le t \le T_i$ is

$$
B_{i-1}(t) = t - T_i + K_{b,i-1} \tag{33}
$$

where $K_{b,i-1}$ is a constant that changes from period to period and is such that $K_{b,i-1} = B_i(T_i)(1-\pi k)$, giving

$$
B_{i-1}(t) = t - T_i + B_i(T_i)(1-\pi k). \tag{34}
$$

Since $B_n(T) = 0$, the solution for $T_n \le t \le T$ is

$$
B_n(t) = (t - T), \tag{35}
$$

the solution for $T_{n-1} \le t \le T_n < T$ is

$$
\begin{aligned}
B_{n-1}(t) &= (t-T_n) + B_n(T_n)(1-\pi k) && (36)\\
&= (t-T_n) + (T_n - T)(1-\pi k),
\end{aligned}
$$

and the solution for $T_{n-2} \le t \le T_{n-1} < T$ is

$$
\begin{aligned}
B_{n-2}(t) &= t - T_{n-1} + B_{n-1}(T_{n-1})(1-\pi k) && (37)\\
&= t - T_{n-1} + ((T_{n-1}-T_n) + (T_n - T)(1-\pi k))(1-\pi k).
\end{aligned}
$$

In this recursive fashion B_i can be determined for all time periods. If all time intervals are $T_i - T_{i-1} = \Delta = (T - T_n)$ for $i = 1$ to n, then we obtain

$$
B_0(t) = t - T_1 + \Delta \textstyle\sum_{i=1}^{n} (1-\pi k)^i \tag{38}
$$

for $t \le T_1$. We notice that $\Delta < 0$. For $n \to \infty$ we also obtain

$$
B_0(t) = t - T_1 + \Delta\left(\frac{1-\pi k}{\pi k}\right). \tag{39}
$$

Then for any generic interval $T_{i-1} \leq t \leq T_i$, to determine C_{i-1} we need to solve $\frac{\partial C_{i-1}}{\partial t} - C_{i-1}b = 0$. The solution is

$$C_{i-1}(t) = K_{c,i-1} \cdot e^{bt} \tag{40}$$

where $K_{c,i-1}$ is a generic constant. Then for time T_{i-1} we impose

$$C_{i-2}(T_{i-1}) = C_{i-1}(T_{i-1}) + \pi k B_{i-1}(T_{i-1}) \tag{41}$$

so that

$$K_{c,i-2} e^{bT_{i-1}} = K_{c,i-1} e^{bT_{i-1}} + \pi k \left(T_{i-1} - T_i + K_{b,i-1}\right) \tag{42}$$

and

$$K_{c,i-2} = K_{c,i-1} + e^{-bT_{i-1}} \pi k \left(T_{i-1} - T_i + K_{b,i}\right). \tag{43}$$

This recursive equation determines $K_{c,i}$ and C_i for all time intervals of the bond life. The recursion starts at the end of the life of the bond. Since $C_n(T) = 0$, $K_{c,n} = 0$ and C_n during the last time interval of the bond life $T_n \leq t \leq T$.

Then by substituting for B_{i-1}, C_{i-1} and D into 25 we can re-write the ODE that A_{i-1} satisfies during the interval $T_{i-1} \leq t \leq T_i$ as

$$\frac{\partial A_{i-1}}{\partial t} + \frac{e^{-a(T-t)} - 1}{a} K_{c,i-1} e^{bt} \rho q \sigma + \left(\frac{e^{-a(T-t)} - 1}{a}\right)^2 \frac{1}{2}\sigma^2 + \left(K_{c,i-1} e^{bt}\right)^2 \frac{1}{2} q^2 + K_{c,i-1} e^{bt} bm = 0. \tag{44}$$

The generic solution to 44 is

$$A_{i-1}(t) = K_{a,i-1} + \sigma^2 e^{-(T-t)} - \frac{\sigma^2}{2} t - \frac{\sigma^2}{4} e^{-2(T-t)} - e^{bt} K_{c,i-1}\left(m - \frac{q\sigma\rho}{b} + q\sigma\rho \frac{e^{t-T}}{b+1}\right) - \frac{K_{c,i-1}^2 q^2}{4b} e^{2bt} \tag{45}$$

where $K_{a,i-1}$ is a generic constant. Since $A_n(T) = 0$, it follows that for $T_n \leq t \leq T$

$$K_{a,n} = -\sigma^2 + \frac{\sigma^2}{2} T + \frac{\sigma^2}{4} + e^{bT} K_{c,n}\left(m - \frac{q\sigma\rho}{b} + \frac{q\sigma\rho}{b+1}\right) + \frac{K_c^2 q^2}{4b} e^{2bT}. \tag{46}$$

It follows that for $T_n \leq t \leq T$ we have

$$\begin{aligned} A_n(t) &= \sigma^2\left(e^{-(T-t)} - 1\right) + \frac{\sigma^2}{2}(T-t) + \frac{\sigma^2}{4}\left(1 - e^{-2(T-t)}\right) \\ &\quad + e^{bT} K_{c,n}\left(m - \frac{q\sigma\rho}{b} + \frac{q\sigma\rho}{b+1}\right) - e^{bt} K_{c,n}\left(m - \frac{q\sigma\rho}{b} + \frac{q\sigma\rho e^{t-T}}{b+1}\right) \\ &\quad + \frac{K_{c,n}^2 q^2}{4b}\left(e^{2bT} - e^{2bt}\right) \end{aligned} \tag{47}$$

with $K_{c,n} = 0$, which gives

$$A_n(t) = \sigma^2 \left(e^{t-T} - 1\right) + \frac{\sigma^2}{2}(T-t) + \frac{\sigma^2}{4}\left(1 - e^{2(t-T)}\right). \tag{48}$$

For $T_{i-2} \leq t \leq T_{i-1}$ we have $C(t) = K_{c,i-2}e^{bt}$ and we impose that $A_{i-2}(T_{i-1}) = A_{i-1}(T_{i-1})$, giving

$$\begin{aligned}
&K_{a,i-2} + \sigma^2 e^{-(T-T_{i-1})} - \frac{\sigma^2}{2}T_{i-1} - \frac{\sigma^2}{4}e^{-2(T-T_{i-1})} - e^{bT_{i-1}}K_{c,i-2}\left(m - \frac{q\sigma\rho}{b} + q\sigma\rho\frac{e^{T_{i-1}-T}}{b+1}\right) \\
&- \frac{K_{c,i-2}^2 q^2}{4b}e^{2bT_{i-1}} = \\
&K_{a,i-1} + \sigma^2 e^{-(T-T_{i-1})} - \frac{\sigma^2}{2}T_{i-1} - \frac{\sigma^2}{4}e^{-2(T-T_{i-1})} - e^{bT_{i-1}}K_{c,i-1}\left(m - \frac{q\sigma\rho}{b} + q\sigma\rho\frac{e^{T_{i-1}-T}}{b+1}\right) \\
&- \frac{K_{c,i-1}^2 q^2}{4b}e^{2bT_{i-1}}
\end{aligned} \tag{49}$$

so that

$$\begin{aligned}
K_{a,i-2} = K_{a,i-1} \quad & + \left(K_{c,i-2} - K_{c,i-1}\right)e^{bT_{i-1}}\left(m - \frac{q\sigma\rho}{b} + q\sigma\rho\frac{e^{T_{i-1}-T}}{b+1}\right) \\
& + \left(K_{c,i-2}^2 - K_{c,i-1}^2\right)\frac{e^{2bT_{i-1}}q^2}{4b}.
\end{aligned} \tag{50}$$

This recursive equation determines $K_{a,i}$ and hence A_i for all time intervals of the bond life. Comparative statics with this model exhibit the qualitative behavior of comparative statics with the model of the previous section. In fact as meeting dates become more and more frequent, the model of this section approaches a model whereby the target rate can change at any time.

The model variants so far analysed allow the reference rate x to turn negative. The model variant of the next section overcomes this problem.

3 The bond pricing model when the target rate is always non-negative

This section presents a different version of the bond pricing model. All other things being equal, we now assume that the risk-neutral process of y is such that

$$dy = b(m-y) + q\sqrt{y}dz_y \tag{51}$$

and $\rho = 0$. Thus we now assume that y follows CIR-type process and that it is uncorrelated with the spread s. These two assumptions guarantee that y and the target interest rate x are non-negative, while model tractability is retained. Moreover the CIR-type process of y offers a way to model yield heteroschedasticity. If the spread s were non-negative, bond yields would also be guaranteed to be non-negative. s would be non-negative if we assumed that $ds = -asdt + \sigma\sqrt{s}dz$ rather than $ds = -asdt + \sigma dz$, but in what follows we retain the latter of these two assumptions, since observed spreads can often negative (see e.g. Piazzesi (2005)).

We now derive the solution for discount bond prices. Under the mentioned assumptions, and if x can change at any time, the absence of arbitrage opportunities implies that V satisfies the equation

$$\frac{\partial V}{\partial t}+\frac{\partial^2 V}{\partial s^2}\frac{1}{2}\sigma^2+\frac{\partial^2 V}{\partial y^2}\frac{1}{2}yq^2-\frac{\partial V}{\partial s}as+\frac{\partial V}{\partial y}b\left(m-y\right)-\left(s+x\right)V+k\left(y-x\right)\left(V_\pi-V\right)=0 \tag{52}$$

subject to the terminal condition $V(T) = 1$. The solution to this equation is such that $V = e^{A+Bx+Ds+Cy}$, $V_\pi = e^{A+B(x+\pi)+Ds+Cy}$ and

$$\frac{\partial A}{\partial t} + D^2\frac{1}{2}\sigma^2 + Cbm = 0 \tag{53}$$
$$\frac{\partial B}{\partial t} - 1 - k\left(e^{B\pi} - 1\right) = 0 \tag{54}$$
$$\frac{\partial D}{\partial t} - Da - 1 = 0 \tag{55}$$
$$\frac{\partial C}{\partial t} + C^2\frac{1}{2}q^2 - Cb + k\left(e^{B\pi} - 1\right) = 0 \tag{56}$$

subject to $A(T) = B(T) = C(T) = D(T) = 0$. This system of ODE's can be quickly solved numerically. If we employ again the approximation $e^{B\pi}-1 \backsimeq B\pi$, the second and the fourth equations in the above system become

$$\frac{\partial B}{\partial t} - 1 - kB\pi = 0 \tag{57}$$
$$\frac{\partial C}{\partial t} + C^2\frac{1}{2}q^2 - Cb + kB\pi = 0 \tag{58}$$

but C and A still need to be computed numerically.

3.1 When the target rate can change just on set dates

We now turn again to considering changes in the target rate that can take place just on dates when central bankers are scheduled to meet, i.e. on dates T_i, with $i = 1, .., n$. Under this and the former assumptions of this section, the absence of arbitrage opportunities implies that V must satisfy the equation

$$\frac{\partial V}{\partial t} + \frac{\partial^2 V}{\partial s^2}\frac{1}{2}\sigma^2 + \frac{\partial^2 V}{\partial y^2}\frac{1}{2}yq^2 - \frac{\partial V}{\partial s}as + \frac{\partial V}{\partial y}b\left(m-y\right) - \left(s+x\right)V = 0 \tag{59}$$

subject to the terminal condition $V(T) = 1$ and to the continuity conditions 21 for all dates T_i. Since the solution to this pricing problem is again of the type $V = e^{A+Bx+Ds+Cy}$, $V_\pi = e^{A+B(x+\pi)+Ds+Cy}$, we can again impose the continuity conditions 24 on any of the dates T_i. In this equation we again assumes that $p(x,y) = k(y-x)$. We could more generally assume that $p(x,y) = k(y-x-s)$, but we do not pursue this. In fact, although this more general assumption would not scupper the closed form solutions that we are going to derive, this generalised assumption would allow x to turn negative. Then the solution for V is such that

$$\frac{\partial A_i}{\partial t} + \frac{D^2\sigma^2}{2} + C_i bm = 0 \tag{60}$$

$$\frac{\partial B_i}{\partial t} - 1 = 0 \tag{61}$$

$$\frac{\partial D}{\partial t} - Da - 1 = 0 \tag{62}$$

$$\frac{\partial C_i}{\partial t} + \frac{C_i^2 q^2}{2} - C_i b = 0 \tag{63}$$

subject to the terminal conditions $A_n(T) = B_n(T) = C_n(T) = D(T) = 0$ and subject again to the continuity conditions $A_{i-1}(T_i) = A_i(T_i)$, $B_{i-1}(T_i) = B_i(T_i)(1-\pi k)$, $C_{i-1}(T_i) = C_i(T_i) + \pi k B_i(T_i)$ for all dates $T_i < T$. During any time interval $[T_{i-1}, T_i]$ the solution for B is the same as in equation 33 and the solution for D is the same as in equation 32. The generic solution for C_i is

$$C_i(t) = 2\frac{b}{q^2}\frac{e^{bt-2bK_{c,i}}}{e^{bt-2bK_{c,i}}-1} \tag{64}$$

where $K_{c,i}$ is a generic constant. For $T_n \leq t \leq T$, $C_n(t) = 0$. For $T_{n-1} \leq t \leq T_n$, we impose that $C_{n-1}(T_n) = 0 + (T_n - T)\pi k$, so that

$$2\frac{b}{q^2}\frac{e^{bT_n-2bK_{c,n-1}}}{e^{bT_n-2bK_{c,n-1}}-1} = 0 + (T_n - T)\pi k \tag{65}$$

giving $K_{c,n-1} = \frac{T_n}{2} - \frac{1}{2b}\left(\ln\frac{q^2(T_n-T)\pi k}{q^2(T_n-T)\pi k - 2b}\right)$. Then for a generic interval $T_{i-2} \leq t \leq T_{i-1}$, we impose that $C_{i-2}(T_{i-1}) = C_{i-1}(T_{i-1}) + B_{i-1}(T_{i-1})\pi k$, so that

$$2\frac{b}{q^2}\frac{e^{bT_{i-1}-2bK_{c,i-2}}}{e^{bT_{i-1}-2bK_{c,i-2}}-1} = 2\frac{b}{q^2}\frac{e^{bT_{i-1}-2bK_{c,i-1}}}{e^{bT_{i-1}-2bK_{c,i-1}}-1} + B_{i-1}(T_{i-1})\pi k \tag{66}$$

giving

$$K_{c,i-2} = \frac{T_{i-1}}{2} + \frac{1}{2b}\ln\left(1 - \frac{1}{\frac{e^{bT_{i-1}-2bK_{c,i-1}}}{e^{bT_{i-1}-2bK_{c,i-1}}-1} + \frac{\pi k q^2(T_{i-1}-T_i+K_{b,i-1})}{2b}}\right) \tag{67}$$

since $B_{i-1}(T_{i-1}) = T_{i-1} - T_i + K_{b,i-1}$. These equations give an iterative solution for $K_{c,i}$ and C_i for all time intervals.

Then we can determine A_{i-1} for the generic interval $T_{i-1} \leq t \leq T_i$ by substituting for C_{i-1} and D into the ODE satisfied by A_{i-1}, so as to obtain

$$\frac{\partial A}{\partial t} + \left(\frac{e^{-a(T-t)} - e^{-\pi k(T-t)}}{a - \pi k} \right)^2 \frac{1}{2}\sigma^2 + 2\frac{b}{q^2} \frac{e^{bt-2bK_{c,i-1}}}{e^{bt-2bK_{c,i-1}} - 1} bm = 0 \tag{68}$$

which has the generic solution

$$\begin{aligned} A_{i-1}(t) \quad = \quad & K_{a,i-1} - \frac{\sigma^2 e^{-(\pi k+1)(T-t)}}{(1-\pi^2 k^2)(\pi k - 1)} - \frac{\sigma^2 e^{-2(T-t)}}{4(\pi k - 1)^2} - \frac{\sigma^2 e^{-2\pi k(T-t)}}{4\pi k(\pi k - 1)^2} \\ & - \frac{4b^2 m}{q^2} K_{c,i-1} - \frac{2bm}{q^2} \ln\left(e^{b(t-2K_{c,i-1})} - 1 \right). \end{aligned} \tag{69}$$

For $T_n \leq t \leq T$, $k = 0$, $C_n(t) = 0$ and $D(t) = \frac{e^{-a(T-t)} - 1}{a}$ so that the ODE satisfied by A_n reduces to

$$\frac{\partial A_n}{\partial t} + \frac{\left(\frac{e^{-a(T-t)}}{a} \right)^2 \sigma^2}{2} = 0 \tag{70}$$

with solution $A_n(t) = K_{a,n} - \frac{\sigma^2}{4} e^{2t-2T}$. Since $A_n(T) = 0$ we deduct that $K_{a,n} = \frac{\sigma^2}{4}$ and

$$A_n(t) = \frac{\sigma^2}{4}\left(1 - e^{-2(T-t)}\right). \tag{71}$$

Given equation 69, for the interval $T_{n-1} \leq t \leq T_n$ we impose

$$A_{n-1}(T_n) = A_n(T_n) \tag{72}$$

giving

$$\begin{aligned} & K_{a,n-1} - \frac{\sigma^2 e^{-(\pi k+1)(T-T_n)}}{(1-\pi^2 k^2)(\pi k - 1)} - \frac{\sigma^2 e^{-2(T-T_n)}}{4(\pi k - 1)^2} - \frac{\sigma^2 e^{-2\pi k(T-T_n)}}{4\pi k(\pi k - 1)^2} \\ & - \frac{4b^2 m K_{c,n-1}}{q^2} - \frac{2bm}{q^2} \ln\left(e^{b(T_n - 2K_{c,n-1})} - 1 \right) = K_{a,n} - \frac{\sigma^2}{4} e^{2T_n - 2T} \end{aligned} \tag{73}$$

and

$$\begin{aligned} K_{a,n-1} \quad = \quad & K_{a,n} - \frac{\sigma^2}{4} e^{2T_n - 2T} + \frac{\sigma^2 e^{-(\pi k+1)(T-T_n)}}{(1-\pi^2 k^2)(\pi k - 1)} + \frac{\sigma^2 e^{-2(T-T_n)}}{4(\pi k - 1)^2} + \frac{\sigma^2 e^{-2\pi k(T-T_n)}}{4\pi k(\pi k - 1)^2} \\ & + \frac{4b^2 m K_{c,n-1}}{q^2} + \frac{2bm}{q^2} \ln\left(e^{b(T_n - 2K_{c,n-1})} - 1 \right). \end{aligned} \tag{74}$$

Given equation 69, for the generic interval $T_{i-2} \leq t \leq T_{i-1}$ we impose

$$A_{i-2}(T_{i-1}) = A_{i-1}(T_{i-1}) \tag{75}$$

giving

$$K_{a,i-2} - \sigma^2 \frac{\exp(T_{i-1} - T - \pi T k + \pi k T_{i-1})}{-\pi^3 k^3 + \pi^2 k^2 + \pi k - 1} - \sigma^2 \frac{e^{2T_{i-1} - 2T}}{4\pi^2 k^2 - 8\pi k + 4} - \sigma^2 \frac{e^{2\pi k T_{i-1} - 2\pi T k}}{4\pi^3 k^3 - 8\pi^2 k^2 + 4\pi k} \tag{76}$$

$$- \frac{4b^2 m K_{c,i-2}}{q^2} - \frac{2bm}{q^2} \ln\left(e^{b(T_{i-1} - 2K_{c,i-2})} - 1\right) =$$

$$= K_{a,i-1} - \sigma^2 \frac{\exp(T_{i-1} - T - \pi k T + \pi k T_{i-1})}{-\pi^3 k^3 + \pi^2 k^2 + \pi k - 1} - \sigma^2 \frac{e^{2T_{i-1} - 2T}}{4\pi^2 k^2 - 8\pi k + 4} - \sigma^2 \frac{e^{2\pi k T_{i-1} - 2\pi T k}}{4\pi^3 k^3 - 8\pi^2 k^2 + 4\pi k}$$

$$- \frac{4b^2 m}{q^2} K_{c,i-1} - \frac{2bm}{q^2} \ln\left(e^{b(T_{i-1} - 2K_{c,i-1})} - 1\right)$$

and

$$K_{a,i-2} = K_{a,i-1} + \frac{4b^2 m}{q^2}(K_{c,i-2} - K_{c,i-1}) + \frac{2bm}{q^2} \ln\left(\frac{e^{b(T_{i-1} - 2K_{c,i-2})} - 1}{e^{b(T_{i-1} - 2K_{c,i-1})} - 1}\right). \tag{77}$$

These equations enable us to determine $K_{a,i}$ and A_i iteratively for any time interval. Again comparative statics with this model exhibit qualitative behavior similar to that of the models of the previous section.

3.2 Generalisations

It is worth highlighting that various assumptions underlying the models of the last two sections can be relaxed while retaining closed form solutions for bond prices of essentially the same type as the ones that have just been shown. For example similar solutions are possible if we assume that on central bankers's meeting dates x may change not only to $x \pm \pi$, but also to $x \pm 2\pi$ and so on. Or we could assume that $p(x, y) = g + k(y - x - s)$ when x can just change on meeting dates. Or we could assume that x could change at any time with constant intensity $\lambda = g$ as well as on the dates when central bankers are due to meet. Or we could assume frequent central bankers' meetings and use the above formulae to approximate a situation whereby x could change at any time. Or we could make the parameters k and π depend on the meeting date T_i to reflect detailed expectations of future target rate changes.

4 Model calibration and estimation

The above bond pricing models can be conveniently calibrated and estimated using observed yields. In fact bond prices are exponential affine in the state variables x, s, and y. Upon calibrating or estimating the model, a major simplification is due to the fact r and x can be readily observed and that we can also immediately find $s = r - x$. y can be inferred by "inverting" observed yields, which is a well known virtue of affine models. For example, if we perfectly observe the 5 year discount bond yield $yield(5)$, then we can infer y on any given date as

$$y = \frac{-5 \cdot yield(5) - A(5) - B(5)\,x - D(5)\,s}{C(5)}. \tag{78}$$

Then calibration can be done by using the whole range of observed yields on a set of dates and the model parameters $\rho, q, \sigma, a, b, m, k$ can be chosen to minimise the sum of squared pricing errors, i.e. the sum of the squared differences between observed yields and the corresponding yields predicted by the model for all dates and all maturities.

The possibility to observe r, x and s and to infer y offers important advantages also in model estimation. As in the past literature that estimates and tests affine models, estimation of the above models can employ a maximum likelihood, simulated maximum likelihood, Generalised Method of Moments or the Kalman filter.

5 Conclusions

This paper has proposed an affine term structure model that hinges on the observed target interest rate set by the central bank. The model develops the innovative approach proposed by Piazzesi (2005): the short-term interest rate is equal to the target rate set by the central bank plus a spread. Like in Piazzesi bond yields are driven by the intensity that determines the probability that the central bank will rise or cut the target rate. Thus the model describes how the yield curve is driven by market expectations about the future rate setting decisions of the central bank. Unlike in Piazzesi (2005) accurate approximations to the bond pricing equation provide new closed form solutions that require no numerical integration for zero coupon bond prices. Unlike in Piazzesi the target interest rate can be constrained to be non-negative, and negative intensities have a convenient interpretation and does not complicate estimation. Yields, especially long term ones, decrease when the central bank is expected to operate more frequent and/or larger average future changes in the target interest rate. The model lends itself to easy calibration and estimation.

A Appendix

A.1 Derivation of the closed form solution when the target rate can change at any time

This appendix determines B, C, D and A when the target rate can change at any time under the assumptions of section 2. To determine B we solve $\frac{\partial B}{\partial t} = 1 + \pi k B$.

The solution is $B(t) = \frac{1}{\pi k}\left(\pi k e^{\pi k t} Q_b - 1\right)$ where Q_b is an arbitrary constant. To determine Q_b we impose the terminal condition $B(T) = 0$, which implies $Q_b = \frac{1}{\pi k} e^{-\pi k T}$, so that $B(t) = \frac{1}{\pi k}\left(e^{-\pi k (T-t)} - 1\right)$.

To determine C we solve $\frac{\partial C}{\partial t} - C \cdot b + e^{-\pi k (T-t)} - 1 = 0$. The solution is

$$C(t) = \frac{1}{b^2 e^{\pi T k} - \pi b k e^{\pi T k}}\left(b e^{\pi k t} - b e^{\pi T k} + \pi k e^{\pi T k}\right) + e^{bt} Q_c \tag{79}$$

where Q_c is an arbitrary constant. To determine Q_c we impose the terminal condition $C(T) = 0$, which implies $Q_c = -\frac{1}{b^2 - \pi b k} \pi k e^{-bT}$, so that

$$C(t) = \frac{b\left(e^{-\pi k (T-t)} - 1\right) + \pi k\left(1 - e^{-b(T-t)}\right)}{b^2 - \pi b k}. \tag{80}$$

To determine D we solve $\frac{\partial D}{\partial t} - Da - e^{-\pi k (T-t)} = 0$. The solution is

$$D(t) = e^{at} Q_d - \frac{e^{\pi k t}}{a e^{\pi T k} - \pi k e^{\pi T k}} \tag{81}$$

where Q_d is an arbitrary constant. To determine Q_d we impose the terminal condition $D(T) = 0$, which implies $Q_d = \frac{e^{-aT}}{a - \pi k}$, so that

$$D(t) = \frac{e^{-a(T-t)} - e^{-\pi k (T-t)}}{a - \pi k}. \tag{82}$$

Then substituting B, C and D into the ODE satisfied by A we obtain

$$\begin{aligned}
&\frac{\partial A}{\partial t} + \left(\frac{e^{-a(T-t)} - e^{-\pi k (T-t)}}{a - \pi k}\right)\left(\frac{b\left(e^{-\pi k (T-t)} - 1\right) + \pi k\left(1 - e^{-b(T-t)}\right)}{b^2 - \pi b k}\right)\rho q \sigma \\
&+ \left(\frac{e^{-a(T-t)} - e^{-\pi k (T-t)}}{a - \pi k}\right)^2 \frac{1}{2}\sigma^2 + \left(\frac{b\left(e^{-\pi k (T-t)} - 1\right) + \pi k\left(1 - e^{-b(T-t)}\right)}{b^2 - \pi b k}\right)^2 \frac{1}{2} q^2 \\
&+ \frac{b\left(e^{-\pi k (T-t)} - 1\right) + \pi k\left(1 - e^{-b(T-t)}\right)}{b^2 - \pi b k} b m = 0
\end{aligned} \tag{83}$$

and this ODE has the solution

$$\begin{aligned}
A(t) \quad = \quad & Q_a - \frac{1}{2b^2} t\left(q^2 - 2b^2 m\right) + \exp\left(at - Ta - \pi T k + \pi k t\right) K_1 \\
& - \frac{e^{\pi k t - \pi T k}}{\pi^2 b^2 k^2 - \pi a b^2 k - \pi^3 b k^3 + \pi^2 a b k^2} K_2 - \exp\left(bt - Tb - \pi T k + \pi k t\right) K_3 \\
& - \frac{e^{bt - Tb}}{b^4 - \pi b^3 k}\left(\pi k q^2 - \pi b^2 k m\right) - e^{2\pi k t - 2\pi T k} K_4 \\
& - e^{2at - 2Ta} \frac{\sigma^2}{4a^3 - 8\pi a^2 k + 4\pi^2 a k^2} - e^{2bt - 2Tb} \frac{\pi^2 k^2 q^2}{4b^5 - 8\pi b^4 k + 4\pi^2 b^3 k^2} \\
& + e^{at - Ta} \frac{q \sigma \rho}{a^2 b e^{at - Ta} - \pi a b k e^{at - Ta}} + \exp\left(at - Tb - Ta + bt\right) K_5
\end{aligned} \tag{84}$$

with

$$
\begin{aligned}
K_1 &= \frac{b\sigma^2 - \pi k\sigma^2 - aq\sigma\rho + \pi kq\sigma\rho}{-\pi a^3 k + ba^3 + \pi^2 a^2 k^2 - \pi ba^2 k + \pi^3 ak^3 - \pi^2 bak^2 - \pi^4 k^4 + \pi^3 bk^3} && (85)\\
K_2 &= aq^2 - \pi kq^2 - ab^2 m - bq\sigma\rho + \pi b^2 km + \pi kq\sigma\rho && (86)\\
K_3 &= \frac{\pi^2 k^2 q^2 - \pi^2 \sigma\rho k^2 q - \pi akq^2 + \pi b\sigma\rho kq}{-\pi b^4 k + ab^4 + \pi^2 b^3 k^2 - \pi ab^3 k + \pi^3 b^2 k^3 - \pi^2 ab^2 k^2 - \pi^4 bk^4 + \pi^3 abk^3} && (87)\\
K_4 &= \frac{a^2 q^2 - 2\rho abq\sigma - 2\pi akq^2 + 2\pi\rho akq\sigma + b^2\sigma^2 + 2\pi\rho bkq\sigma - 2\pi bk\sigma^2 + \pi^2 k^2 q^2 - 2\pi^2 \rho k^2 q\sigma + \pi^2 k^2 \sigma^2}{4\pi a^2 b^2 k - 8\pi^2 a^2 bk^2 + 4\pi^3 a^2 k^3 - 8\pi^2 ab^2 k^2 + 16\pi^3 abk^3 - 8\pi^4 ak^4 + 4\pi^3 b^2 k^3 - 8\pi^4 bk^4 + 4\pi^5 k^5} && (88)\\
K_5 &= \frac{\pi kq\sigma\rho}{a^2 b^2 - \pi a^2 bk + ab^3 - 2\pi ab^2 k + \pi^2 abk^2 - \pi b^3 k + \pi^2 b^2 k^2}. && (89)
\end{aligned}
$$

To determine Q_a we impose that $A(T) = 0$ so that

$$
\begin{aligned}
Q_a &= \frac{1}{2b^2} T\left(q^2 - 2b^2 m\right) - K_1 + K_2 + K_3 + \frac{\pi kq^2 - \pi b^2 km}{b^4 - \pi b^3 k} + K_4 - K_5 && (90)\\
&\quad + \frac{\sigma^2}{4a^3 - 8\pi a^2 k + 4\pi^2 ak^2} + \frac{\pi^2 k^2 q^2}{4b^5 - 8\pi b^4 k + 4\pi^2 b^3 k^2} - \frac{q\sigma\rho}{a^2 b e^{at-Ta} - \pi abke^{at-Ta}}
\end{aligned}
$$

and

$$
\begin{aligned}
&A(t) = (T-t)\frac{q^2 - 2b^2 m}{2b^2} + K_1\left(e^{at-Ta-\pi Tk+\pi kt} - 1\right) && (91)\\
&+ K_2\left(1 - \frac{e^{\pi kt - \pi Tk}}{\pi^2 b^2 k^2 - \pi ab^2 k - \pi^3 bk^3 + \pi^2 abk^2}\right) + K_3\left(1 - e^{bt-Tb-\pi Tk+\pi kt}\right)\\
&+ \frac{\pi kq^2 - \pi b^2 km}{b^4 - \pi b^3 k}\left(1 - e^{bt-Tb}\right) + K_4\left(1 - e^{2\pi kt - 2\pi Tk}\right) + \frac{\sigma^2\left(1 - e^{2at-2Ta}\right)}{4a^3 - 8\pi a^2 k + 4\pi^2 ak^2}\\
&+ \frac{\pi^2 k^2 q^2\left(1 - e^{2bt-2Tb}\right)}{4b^5 - 8\pi b^4 k + 4\pi^2 b^3 k^2} + \frac{q\sigma\rho\left(e^{at-Ta} - 1\right)}{a^2 b e^{at-Ta} - \pi abke^{at-Ta}} + K_5\left(e^{at-Tb-Ta+bt} - 1\right).
\end{aligned}
$$

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